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Darboux–Bäcklund solutions of SL(p, q) KP-KdV hierarchies, constrained generalized Toda lattices, and two-matrix string model

H. Aratyn [a,1], E. Nissimov [b,c,2,3], S. Pacheva [b,c,2,4]

[a] Department of Physics, University of Illinois at Chicago, 845 W. Taylor Street, Chicago, IL 60607-7059, USA
[b] Institute of Nuclear Research and Nuclear Energy, Boul. Tsarigradsko Chausee 72, 1784 Sofia, Bulgaria
[c] Department of Physics, Ben-Gurion University of the Negev, Box 653, 84105 Beer Sheva, Israel

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Abstract

We present a unifying description of the graded SL(p, q) KP-KdV hierarchies, including the Wronskian construction of their τ-functions as well as the coefficients of the pertinent Lax operators, obtained via successive applications of special Darboux–Bäcklund transformations. The emerging Darboux–Bäcklund structure is identified as a constrained generalized Toda lattice system relevant for the two-matrix string model. It allows simple derivation of the n-soliton solutions of the unconstrained KP system. Also, the exact Wronskian solution for the two-matrix model partition function is found.

1. Introduction

The Kadomtsev–Petviashvili (KP) hierarchy of integrable soliton nonlinear evolution equations [1] is among the most important physically relevant integrable systems. One of the main reasons for the interest in the KP hierarchy in the last few years originates from its deep connection with the statistical-mechanical models of random matrices ((multi-)matrix models) providing nonperturbative discretized formulation of string theory [2]. Most of the studies in the latter area follow the ideas of the original papers [3], where the integrability structures arise only after taking the continuum *double-scaling* limit. There exists, however, an alternative efficient approach [4] for extracting continuum differential integrable hierarchies from multi-matrix string models even *before* taking the continuum limit. More precisely, it is various reductions of the full KP hierarchy (*constrained* KP hierarchies) which play the major rôle in the latter context.

[1] Work supported in part by the US Department of Energy under contract DE-FG02-84ER40173. E-mail: aratyn@uic.edu.
[2] Supported in part by Bulgarian NSF grant Ph-401.
[3] E-mail: emil@bgearn.bitnet, emil@bgumail.bgu.ac.il.
[4] E-mail: svetlana@bgearn.bitnet, svetlana@bgumail.bgu.ac.il.

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On the other hand, constrained KP hierarchies arise also naturally in a purely solitonic context as shown below in Section 6.2 (see also Ref. [5] and references therein).

It is the aim of the present note to study various properties and provide exact solutions for a specific class of constrained KP hierarchies – the graded $SL(p,q)$ KP-KdV hierarchies, which are intimately related with the two-matrix string model (which is the most physically relevant one). The following main results are contained in the sequel:

(i) We establish the equivalence between conventional "symmetry"-constrained KP hierarchies [5] and multi-boson reductions of the full KP hierarchy [6,7], also known as graded $SL(p,q)$-type KP-KdV hierarchies [8,9], which appear in two-matrix models of string theory [4]. In particular, we provide the explicit Miura map relating the above hierarchies.

(ii) Explicit exact solutions are found for $SL(p,q)$ KP-KdV integrable systems, including eigenfunctions and τ-functions, via special Darboux–Bäcklund (DB) transformations.

(iii) We establish the equivalence between the set of successive DB transformations on the $SL(p,1)$ KP-KdV system and the equations of motion of a *constrained* generalized Toda lattice model, which embodies the integrability structure of two-matrix string models.

(iv) As a byproduct of (iii) we obtain the exact solution for this constrained Toda lattice system under specific initial conditions, relevant in the context of the two-matrix string model, and derive the exact expression for the partition function of the latter.

(v) The present DB formalism provides a simple systematic way to obtain the n-soliton solutions for the full (unconstrained) KP system.

2. Background on generalized KP-KdV hierarchies and Darboux–Bäcklund transformations

2.1. Constrained KP hierarchies

We shall consider the general class of constrained KP Lax operators with higher purely-differential part [5], also known as N-generalized two-boson KP Lax operators [10] [5],

$$L = L_+ + \sum_{i=1}^{N} \Phi_i D^{-1} \Psi_i \equiv D^r + \sum_{l=0}^{r-2} u_l D^l + \sum_{i=1}^{N} a_i (D - b_i)^{-1}, \tag{1}$$

$$L_+ \equiv D^r + \sum_{l=0}^{r-2} u_l D^l, \qquad \Phi_i = a_i \exp\left(\int b_i\right), \qquad \Psi_i = \exp\left(-\int b_i\right). \tag{2}$$

One can also define an alternative consistent Poisson reduction of the standard KP hierarchy based on the pseudo-differential Lax operators of the form [4,6,7],

$$L_N = L_+ + \sum_{i=1}^{N} r_i \prod_{k=i}^{N} D^{-1} q_k = L_+ + \sum_{i=1}^{N} A_i^{(N)} \prod_{k=i}^{N} (D - B_k^{(N)})^{-1}, \tag{3}$$

$$r_i = A_i \exp\left(\int B_i\right), \qquad q_N = \exp\left(-\int B_N\right), \qquad q_j = \exp\left(\int (B_j - B_{j+1})\right),$$
$$j = 1, \ldots, N-1, \tag{4}$$

[5] In order to avoid confusion, D will denote the differential operator in the sense of pseudo-differential calculus, whereas the derivative of a function will be denoted as $\partial_x f$.

which we will call multi-boson reduction of the full KP Lax operator. The above multi-boson reductions of the full KP Lax operators (3) define the generalized graded $SL(r+N,N)$ KP-KdV hierarchies pertinent to the string two-matrix models (cf. Refs. [9,11]).

In Ref. [10] these two formulations of the constrained KP hierarchy have been related via successive DB similarity transformations. Below in Section 3 we will establish their complete equivalence showing how the pseudo-differential Lax operators (1) and (3) can be rewritten into each other via a generalized Miura transformation. Due to this result we can limit ourselves to study DB transformations within the framework of the constrained KP hierarchy defined by N-generalized two-boson KP Lax operators as in (1).

2.2. On the DB transformations of the N-generalized two-boson KP Lax operators

The general form of a DB transformation on the N-generalized two-boson KP Lax operator (1) reads [12,13]

$$\widetilde{L} = \chi D \chi^{-1}\left(L_+ + \sum_{i=1}^{N} \Phi_i D^{-1}\Psi_i\right)\chi D^{-1}\chi^{-1} \equiv \widetilde{L}_+ + \widetilde{L}_-, \tag{5}$$

$$\widetilde{L}_+ = L_+ + \chi[\partial_x(\chi^{-1}L_+\chi)_{\geqslant 1}D^{-1}]\chi^{-1}, \tag{6}$$

$$\widetilde{L}_- = \widetilde{\Phi}_0 D^{-1}\widetilde{\Psi}_0 + \sum_{i=1}^{N}\widetilde{\Phi}_i D^{-1}\widetilde{\Psi}_i, \tag{7}$$

$$\widetilde{\Phi}_0 = \chi\left(\partial_x(\chi^{-1}L_+\chi) + \sum_{i=1}^{N}[\partial_x(\chi^{-1}\Phi_i)\partial_x^{-1}(\Psi_i\chi) + \Phi_i\Psi_i]\right) \equiv (\chi D\chi^{-1}L)\chi, \tag{8}$$

$$\widetilde{\Psi}_0 = \chi^{-1}, \qquad \widetilde{\Phi}_i = \chi\partial_x(\chi^{-1}\Phi_i), \qquad \widetilde{\Psi}_i = -\chi^{-1}\partial_x^{-1}(\Psi_i\chi), \tag{9}$$

where all functions involved are (adjoint) eigenfunctions of L (1), i.e., they satisfy

$$\frac{\partial}{\partial t_n} f = L_+^{n/r} f, \qquad f = \chi,\ \Phi_i, \qquad \frac{\partial}{\partial t_n}\Psi_i = -L_+^{*n/r}\Psi_i. \tag{10}$$

Let us particularly stress that the above eigenfunctions *are not* Baker–Akhiezer eigenfunctions of L from (1), unlike the construction in Ref. [12].

We are interested in the special case when χ coincides with one of the original eigenfunctions of L, e.g. $\chi = \Phi_1$. Then $\widetilde{\Phi}_1 = 0$ and the DB transformation (5) preserves the N-generalized two-boson form (1) of the Lax operators involved, i.e., it becomes an *auto*-Bäcklund transformation. Applying successive DB transformations in this case yields

$$L^{(k)} = T^{(k-1)}L^{(k-1)}(T^{(k-1)})^{-1} = (L^{(k)})_+ + \sum_{i=1}^{N}\Phi_i^{(k)}D^{-1}\Psi_i^{(k)}, \qquad T^{(k)} \equiv \Phi_1^{(k)}D(\Phi_1^{(k)})^{-1}, \tag{11}$$

$$\Phi_1^{(k+1)} = (T^{(k)}L^{(k)})\Phi_1^{(k)}, \qquad \Psi_1^{(k+1)} = (\Phi_1^{(k)})^{-1}, \qquad k = 0,1,\dots, \tag{12}$$

$$\Phi_i^{(k+1)} = T^{(k)}\Phi_i^{(k)} \equiv \Phi_1^{(k)}\partial_x[(\Phi_1^{(k)})^{-1}\Phi_i^{(k)}], \tag{13}$$

$$\Psi_i^{(k+1)} = -(\Phi_1^{(k)})^{-1}\partial_x^{-1}(\Psi_i^{(k)}\Phi_1^{(k)}), \qquad i = 2,\dots,N. \tag{14}$$

Using the first identity (11), i.e., $L^{(k+1)}T^{(k)} = T^{(k)}L^{(k)}$, one can rewrite (12) in the form

$$\Phi_1^{(k)} = T^{(k-1)}T^{(k-2)}\dots T^{(0)}[(L^{(0)})^k\Phi_1^{(0)}], \tag{15}$$

whereas

$$\Phi_i^{(k)} = T^{(k-1)}T^{(k-2)}\dots T^{(0)}\Phi_i^{(0)}, \qquad i = 2,\dots,N. \tag{16}$$

Finally, for the coefficient of the next-to-leading differential term in (1) $u_{r-2} = r\,\mathrm{Res}\,L^{1/r} = r\partial_x^2 \ln\tau$, we easily obtain from (6) (with $\chi = \Phi_1$) its k-step DB-transformed expression,

$$\frac{1}{r}(u_{r-2}^{(k)} - u_{r-2}^{(0)}) = \partial_x^2 \ln\frac{\tau^{(k)}}{\tau^{(0)}} = \partial_x^2 \ln(\Phi_1^{(k-1)}\dots\Phi_1^{(0)}). \tag{17}$$

2.3. Wronskian preliminaries

Firstly we list three basic properties of Wronskian determinants.

(i) The derivative $\mathcal{D}'$ of a determinant $\mathcal{D}$ of order n, whose entries are differentiable functions, can be written as

$$\mathcal{D}' = \mathcal{D}_{(1)} + \mathcal{D}_{(2)} + \dots + \mathcal{D}_{(n)}, \tag{18}$$

where $\mathcal{D}_{(i)}$ is obtained from D by differentiating the entries in the ith row.

(ii) *Jacobi expansion theorem.*

$$W_k(f)W_{k-1} = W_kW_{k-1}'(f) - W_k'W_{k-1}(f), \qquad \text{or} \qquad \partial\left(\frac{W_{k-1}(f)}{W_k}\right) = \frac{W_k(f)W_{k-1}}{W_k^2}, \tag{19}$$

where the Wronskians are $W_k \equiv W_k[\psi_1,\dots,\psi_k]$ and $W_{k-1}(f) \equiv W_k[\psi_1,\dots,\psi_{k-1},f]$. For proof see Ref. [14]. Take a special class of Wronskians $W_n \equiv W_n[\psi,\psi',\dots,\partial^{n-1}\psi]$. Hence, from (19) we get

$$W_nW_n'' - W_n'W_n' = W_nW_{n-1}'(\partial^n\psi) - W_n'W_{n-1}(\partial^n\psi) = W_{n-1}W_{n+1} \to \partial^2\ln W_n = \frac{W_{n+1}W_{n-1}}{W_n^2}. \tag{20}$$

(iii) Iterative composition of Wronskians.

$$T_kT_{k-1}\dots T_1(f) = \frac{W_k(f)}{W_k}, \tag{21}$$

where

$$T_j = \frac{W_j}{W_{j-1}}D\frac{W_{j-1}}{W_j} = D + \left(\ln\frac{W_{j-1}}{W_j}\right)', \qquad W_0 = 1. \tag{22}$$

The proof of (21) follows by simple iteration of (19) (see also the standard references on this subject [14–16]). For future use let us rewrite (21) as

$$(D+v_k)(D+v_{k-1})\dots(D+v_1)f = \frac{W_k(f)}{W_k}, \qquad v_j \equiv \partial_x\ln\frac{W_{j-1}}{W_j}. \tag{23}$$

2.4. DB solutions of two-bose KP system and connection with ordinary Toda lattice

The two-boson KP system defined by the Lax operator $L = D + \Phi D^{-1}\Psi \equiv D + a(D-b)^{-1}$ is the most basic constrained KP structure. We start with the initial "free" Lax operator $L^{(0)} = D$ and perform a DB transformation,

$$L^{(1)} = (\Phi^{(0)}D\Phi^{(0)^{-1}})D(\Phi^{(0)}D^{-1}\Phi^{(0)^{-1}}) = D + [\Phi^{(0)}(\ln\Phi^{(0)})'']D^{-1}(\Phi^{(0)})^{-1}. \tag{24}$$

The construction below is a special application of property (iii) and Eq. (21).

In the case under consideration the relevant formulas for successive DB transformations (12), (11) specialize to

$$L^{(k+1)} = (\Phi^{(k)} D \Phi^{(k)^{-1}}) L^{(k)} (\Phi^{(k)} D^{-1} \Phi^{(k)^{-1}}) = D + \Phi^{(k+1)} D^{-1} \Psi^{(k+1)}, \tag{25}$$

$$\Phi^{(k+1)} = \Phi^{(k)} (\ln \Phi^{(k)})'' + (\Phi^{(k)})^2 \Psi^{(k)}, \qquad \Psi^{(k+1)} = (\Phi^{(k)})^{-1}, \tag{26}$$

whereas Eq. (15) acquires the form (proved easily by induction in k)

$$\Phi^{(k)} = \frac{W_{k+1}[\Phi, \partial\Phi, \dots, \partial^k \Phi]}{W_k[\Phi, \partial\Phi, \dots, \partial^{k-1}\Phi]}, \qquad \text{with } \Phi = \Phi^{(0)}. \tag{27}$$

Introduce now

$$\phi_k = \ln \Phi^{(k)} \to \Phi^{(k)} = \exp(\phi_k), \qquad k = 0, \dots , \tag{28}$$

which allows us to rewrite (26) as a (ordinary one-dimensional) Toda lattice equation

$$\partial^2 \phi_k = \exp(\phi_{k+1} - \phi_k) - \exp(\phi_k - \phi_{k-1}). \tag{29}$$

Introduce now new objects ψ_n as follows,

$$\phi_n = \psi_{n+1} - \psi_n \to \psi_n = \ln W_n[\Phi, \partial\Phi, \dots, \partial^{n-1}\Phi]. \tag{30}$$

From Eq. (20) we find immediately an equation for ψ_n,

$$\partial^2 \psi_n = \partial^2 \ln W_n = \exp(\psi_{n+1} + \psi_{n-1} - 2\psi_n), \tag{31}$$

with $\psi_n = 0$ for $n \leqslant 0$. We recognize in the right-hand side of (31) a structure of the Cartan matrix for A_n. Leznov considered such an equation with Wronskian solution (in two dimensions) in Ref. [17].

Hence, the solutions of the (ordinary one-dimensional) Toda lattice equations, with boundary conditions $\psi_n = 0$ for $n \leqslant 0$, reproduce the DB solutions of ordinary two-boson KP hierarchy (27) upon taking into account that $\Phi = \Phi^{(0)} = \exp(\phi_0) = \exp(\psi_1)$.

3. Equivalence between N-generalized two-boson and $2N$-boson KP hierarchies

First, let us consider the simplest nontrivial case $N = 2$ in (1). Applying the simple identity

$$\begin{aligned} \phi D^{-1} \psi &= \phi\psi(\chi^{-1} D^{-1} \chi) - \phi D^{-1} \left[\chi \left(\partial_x \frac{\psi}{\chi} \right) \right] (\chi^{-1} D^{-1} \chi) \\ &= \phi\psi(\chi^{-1} D^{-1} \chi) - \phi \frac{W[\chi, \psi]}{W[\chi]} \left(\frac{W[\chi]}{W[\chi, \psi]} D^{-1} \frac{W[\chi, \psi]}{W[\chi]} \right) (\chi^{-1} D^{-1} \chi), \end{aligned} \tag{32}$$

for arbitrary functions ϕ, ψ, χ, where in the second equality Wronskian identity (21) was used, we obtain

$$L = L_+ + \Phi_1 D^{-1} \Psi_1 + \Phi_2 D^{-1} \Psi_2 = L_+ + A_2^{(2)} (D - B_2^{(2)})^{-1} + A_1^{(2)} (D - B_1^{(2)})^{-1} (D - B_2^{(2)})^{-1}, \tag{33}$$

$$A_2^{(2)} = \Phi_1 \Psi_1 + \Phi_2 \Psi_2, \qquad B_2^{(2)} = -\partial_x \ln \Psi_2, \tag{34}$$

$$A_1^{(2)} = -\Phi_1 \frac{W[\Psi_2, \Psi_1]}{W[\Psi_2]}, \qquad B_1^{(2)} = -\partial_x \ln \frac{W[\Psi_2, \Psi_1]}{W[\Psi_2]}. \tag{35}$$

Using successively the same type of identity as (32), together with (21), e.g., for arbitrary functions ϕ, ψ, χ, ω,

$$\phi D^{-1}\psi = \phi\psi(\chi^{-1}D^{-1}\chi) - \phi\frac{W[\chi,\psi]}{W[\chi]}\left(\frac{W[\chi]}{W[\chi,\omega]}D^{-1}\frac{W[\chi,\omega]}{W[\chi]}\right)(\chi^{-1}D^{-1}\chi)$$
$$+\phi\frac{W[\chi,\omega,\psi]}{W[\chi,\omega]}\left(\frac{W[\chi,\omega]}{W[\chi,\omega,\psi]}D^{-1}\frac{W[\chi,\omega]}{W[\chi]}\right)\left(\frac{W[\chi]}{W[\chi,\omega]}D^{-1}\frac{W[\chi,\omega]}{W[\chi]}\right)(\chi^{-1}D^{-1}\chi), \tag{36}$$

we can prove by induction in N that the N-generalized two-boson KP Lax operator can be transformed into the $2N$-boson KP Lax operator,

$$L = D^r + \sum_{l=0}^{r-2} u_l D^l + \sum_{i=1}^{N} \Phi_i D^{-1}\Psi_i \equiv L_+ + \sum_{i=1}^{N} a_i(D-b_i)^{-1} \tag{37}$$

$$= L_+ + \sum_{i=1}^{N} A_i^{(N)}(D-B_i^{(N)})^{-1}(D-B_{i+1}^{(N)})^{-1}\dots(D-B_N^{(N)})^{-1} \tag{38}$$

upon the following change of variables, i.e., generalized Miura transformation,

$$A_k^{(N)} = (-1)^{N-k}\sum_{s=1}^{k}\Phi_s\frac{W[\Psi_N,\dots,\Psi_{k+1},\Psi_s]}{W[\Psi_N,\dots,\Psi_{k+1}]}, \tag{39}$$

$$B_k^{(N)} = -\partial_x \ln\frac{W[\Psi_N,\dots,\Psi_{k+1},\Psi_k]}{W[\Psi_N,\dots,\Psi_{k+1}]}. \tag{40}$$

Let us now illustrate the equivalence between (37) and (38) in the inverse direction. To this end it is more convenient to use the (r,q) form of (3). Let us define the quantity

$$Q_{k,i} \equiv (-1)^{i-k}\int q_i\int q_{i-1}\int\dots\int q_k\,(\mathrm{d}x')^{i-k+1}, \qquad 1 \leqslant k \leqslant i \leqslant N. \tag{41}$$

Then using $D^{-1}Q_{1,i-1}q_i = D^{-1}Q_{1,i}D - Q_{1,i}$ we obtain from Eq. (3)

$$L_N = L_+ + \sum_{i=2}^{N} r_i^{(1)}\prod_{k=i}^{N} D^{-1}q_k + r_1 D^{-1}(-Q_{1,N-1}q_N), \tag{42}$$

where

$$r_i^{(1)} \equiv r_i + r_1 Q_{1,i-1}, \qquad i = 2,\dots,N. \tag{43}$$

The above process can be continued to yield expression (1) with

$$\Phi_i = r_i + \sum_{k=1}^{i-1} r_k \sum_{s_{i-k-1}=s_{i-k-2}+1}^{i-k}\cdots\sum_{s_2=s_1+1}^{i-k}\sum_{s_1=1}^{i-k} Q_{k,i-s_{i-k-1}-1}Q_{i-s_{i-k-1},i-s_{i-k-2}-1}\cdots Q_{i-s_2,i-s_1-1}Q_{i-s_1,i-1},$$
$$1 \leqslant i \leqslant N, \tag{44}$$

$$\Psi_N = q_N, \qquad \Psi_i = (-1)^{N-i}q_N\int q_{N-1}\int\dots\int q_i\,(\mathrm{d}x')^{N-i}, \qquad 1 \leqslant i \leqslant N-1. \tag{45}$$

4. Exact solutions of SL(p, q) KP-KdV via DB transformations

With the Wronskian identities from Section 2.3, we can now represent the k-step DB transformation (15)–(17) in terms of Wronskian determinants involving only the coefficient functions of the "initial" Lax operator,

$$L^{(0)} = D^r + \sum_{l=0}^{r-2} u_l^{(0)} D^l + \sum_{i=1}^{N} \Phi_i^{(0)} D^{-1} \Psi_i^{(0)}. \tag{46}$$

Indeed, using identity (21) and defining

$$(L^{(0)})^k \Phi_1^{(0)} \equiv \chi^{(k)}, \qquad k = 1, 2, \dots , \tag{47}$$

we arrive at the following general result:

Proposition. The k-step DB-transformed eigenfunctions and the τ-function (15)–(17) of the SL$(r+N, N)$ KP-KdV system (37) for arbitrary initial $L^{(0)}$ (46) are given by

$$\Phi_1^{(k)} = \frac{W_{k+1}[\Phi_1^{(0)}, \chi^{(1)}, \dots, \chi^{(k)}]}{W_k[\Phi_1^{(0)}, \chi^{(1)}, \dots, \chi^{(k-1)}]}, \tag{48}$$

$$\Phi_j^{(k)} = \frac{W_{k+1}[\Phi_1^{(0)}, \chi^{(1)}, \dots, \chi^{(k-1)}, \Phi_j^{(0)}]}{W_k[\Phi_1^{(0)}, \chi^{(1)}, \dots, \chi^{(k-1)}]}, \qquad j = 2, \dots, N, \tag{49}$$

$$\tau^{(k)} = W_k[\Phi_1^{(0)}, \chi^{(1)}, \dots, \chi^{(k-1)}]\tau^{(0)}, \tag{50}$$

where $\tau^{(0)}$, $\tau^{(k)}$ are the τ-functions of $L^{(0)}$, $L^{(k)}$, respectively, and $\chi^{(i)}$ is given by (47).

As an example let us consider the SL(3, 1) KP-KdV Lax operator, i.e., $r = 2, N = 1$ in (1) (the latter is pertinent to the simplest nontrivial string two-matrix model [11]),

$$L = D^2 + u + A(D-B)^{-1} = D^2 + u + \Phi D^{-1} \Psi. \tag{51}$$

From the basic formulas for the successive DB transformations (12), (11), applied to (51), we have

$$L^{(k)} = D^2 + u^{(k)} + \Phi^{(k)} D^{-1} \Psi^{(k)}, \tag{52}$$

$$L^{(k)} \to L^{(k+1)} = T^{(k)} L^{(k)} (T^{(k)})^{-1}, \qquad T^{(k)} = \Phi^{(k)} D (\Phi^{(k)})^{-1}, \tag{53}$$

$$u^{(k)} = 2\,\mathrm{Res}\, L^{1/2} \equiv 2\partial_x^2 \ln \tau^{(k)} = 2\partial_x^2 \ln(\Phi^{(k-1)} \dots \Phi^{(0)}), \tag{54}$$

$$\Phi^{(k)} \equiv A^{(k)} \exp\left(\int B^{(k)}\right) = \Phi^{(k-1)} \left[\partial_x \left(\frac{1}{\Phi^{(k-1)}} \partial_x^2 \Phi^{(k-1)} + 2\partial_x^2 \ln(\Phi^{(k-2)} \dots \Phi^{(0)})\right) + \frac{\Phi^{(k-1)}}{\Phi^{(k-2)}}\right], \tag{55}$$

$$\Psi^{(k)} \equiv \exp\left(-\int B^{(k)}\right) = (\Phi^{(k-1)})^{-1}, \tag{56}$$

$$u^{(0)} = 0, \qquad \Psi^{(0)} = 0, \qquad \Phi^{(0)} = \int_\Gamma \frac{\mathrm{d}\lambda}{2\pi} c(\lambda) \exp[\xi(\lambda, \{t\})], \qquad \xi(\lambda, \{t\}) \equiv \lambda x + \sum_{j \geqslant 2} \lambda^j t_j, \tag{57}$$

where $\Phi^{(0)}$ in (57) is an arbitrary eigenfunction of the "free" $L^{(0)} = D^2$ (the contour Γ in the complex λ-plane is chosen such that the generalized Laplace transform of $c(\lambda)$ is well defined).

As a corollary from the above proposition, we get in the case of (52),

$$\Phi^{(k)} = T^{(k-1)}\dots T^{(0)}(\partial_x^{2k}\Phi^{(0)}) = \frac{W[\Phi^{(0)},\partial_x^2\Phi^{(0)},\dots,\partial_x^{2k}\Phi^{(0)}]}{W[\Phi^{(0)},\partial_x^2\Phi^{(0)},\dots,\partial_x^{2(k-1)}\Phi^{(0)}]}, \tag{58}$$

$$\tau^{(k)} = W[\Phi^{(0)},\partial_x^2\Phi^{(0)},\dots,\partial_x^{2(k-1)}\Phi^{(0)}]. \tag{59}$$

Substituting (58), (59) into (54)–(56) we obtain the following explicit solutions for the coefficient functions of (51),

$$u^{(n)} = 2\partial_x^2 \ln W[\Phi^{(0)},\partial_x^2\Phi^{(0)},\dots,\partial_x^{2(n-1)}\Phi^{(0)}], \tag{60}$$

$$B^{(n)} = \partial_x \ln\left(\frac{W[\Phi^{(0)},\partial_x^2\Phi^{(0)},\dots,\partial_x^{2(n-1)}\Phi^{(0)}]}{W[\Phi^{(0)},\partial_x^2\Phi^{(0)},\dots,\partial_x^{2(n-2)}\Phi^{(0)}]}\right), \tag{61}$$

$$A^{(n)} = \frac{W[\Phi^{(0)},\partial_x^2\Phi^{(0)},\dots,\partial_x^{2n}\Phi^{(0)}]\,W[\Phi^{(0)},\partial_x^2\Phi^{(0)},\dots,\partial_x^{2(n-2)}\Phi^{(0)}]}{(W[\Phi^{(0)},\partial_x^2\Phi^{(0)},\dots,\partial_x^{2(n-1)}\Phi^{(0)}])^2}. \tag{62}$$

Similarly, in the more general case of the $SL(r+1,1)$ KP-KdV Lax operator for arbitrary finite r,

$$L = D^r + \sum_{l=0}^{r-2} u_l D^l + \Phi D^{-1}\Psi, \tag{63}$$

which defines the integrable hierarchy corresponding to the general string two-matrix model (cf. Refs. [9,11]), the generalizations of (58) and (59) read

$$\Phi^{(k)} = T^{(k-1)}\dots T^{(0)}(\partial_x^{k\cdot r}\Phi^{(0)}) = \frac{W[\Phi^{(0)},\partial_x^r\Phi^{(0)},\dots,\partial_x^{k\cdot r}\Phi^{(0)}]}{W[\Phi^{(0)},\partial_x^r\Phi^{(0)},\dots,\partial_x^{(k-1)\cdot r}\Phi^{(0)}]}, \tag{64}$$

$$\frac{1}{r}u_{r-2}^{(k)} = \operatorname{Res} L^{1/r} = \partial_x^2 \ln \tau^{(k)}, \qquad \tau^{(k)} = W[\Phi^{(0)},\partial_x^r\Phi^{(0)},\dots,\partial_x^{(k-1)\cdot r}\Phi^{(0)}], \tag{65}$$

where $\Phi^{(0)}$ is again given explicitly by (57).

5. Relation to constrained generalized Toda lattices

Here we shall establish the equivalence between the set of successive DB transformations of the $SL(r+1,1)$ KP-KdV system (63),

$$L^{(k+1)} = T^{(k)} L^{(k)} (T^{(k)})^{-1}, \qquad T^{(k)} = \Phi^{(k)} D \Phi^{(k)\,-1}, \tag{66}$$

$$L^{(0)} = D^r + \sum_{l=0}^{r-2} u_l^{(0)} D^l + \Phi^{(0)} D^{-1} \Psi^{(0)}, \tag{67}$$

and the equations of motion of a *constrained* generalized Toda lattice system, underlying the two-matrix string model, which contains, in particular, the *two-dimensional* Toda lattice equations.

For simplicity we shall illustrate the above property on the simplest nontrivial case of $SL(3,1)$ KP-KdV hierarchy (51). We note that Eqs. (54)–(56) (or (60)–(62)) can be cast in the following recurrence form,

$$\partial_x \ln A^{(n-1)} = B^{(n)} - B^{(n-1)}, \tag{68}$$

$$u^{(n)} - u^{(n-1)} = 2\partial_x B^{(n)}, \tag{69}$$

$$A^{(n)} - A^{(n-1)} = \partial_x[(B^{(n)})^2 + \tfrac{1}{2}(u^{(n)} + u^{(n-1)})], \tag{70}$$

with the "initial" conditions (cf. (57))

$$A^{(0)} = B^{(0)} = u^{(0)} = 0, \qquad B^{(1)} \equiv \partial_x \ln \Phi, \tag{71}$$

where Φ is so far an arbitrary function. Now, we can view (68)–(70) as a system of lattice equations for the dynamical variables $A^{(n)}, B^{(n)}, u^{(n)}$ associated with each lattice site n and subject to the boundary conditions

$$A^{(n)} = B^{(n)} = u^{(n)} = 0, \qquad n \leqslant 0. \tag{72}$$

Taking (71) as initial data, one can solve the lattice system (68)–(70) step by step (for $n = 1, 2, \ldots$) and the solution has precisely the form of (60)–(62).

The lattice system (68)–(70) can be identified with the $\tilde{t}_1 \equiv x$ evolution equations of the *constrained* generalized Toda lattice hierarchy defined as follows [4,11],

$$\frac{\partial}{\partial t_r} Q = [Q^r_{(+)}, Q], \qquad \frac{\partial}{\partial t_r} \bar{Q} = [Q^r_{(+)}, \bar{Q}], \qquad r = 1, \ldots, p_1, \tag{73}$$

$$\frac{\partial}{\partial \tilde{t}_s} Q = [Q, \bar{Q}^s_-], \qquad \frac{\partial}{\partial \tilde{t}_s} \bar{Q} = [\bar{Q}, \bar{Q}^s_-], \qquad s = 1, \ldots, p_2, \tag{74}$$

$$-g[Q, \bar{Q}] = \mathbb{1}. \tag{75}$$

Here Q and $\bar{Q}$ are semi-infinite matrices, i.e., with indices running from 0 to ∞, with the following explicit parametrization,

$$Q_{nn} = a_0(n), \qquad Q_{n,n+1} = 1, \qquad Q_{n,n-k} = a_k(n), \qquad k = 1, \ldots, p_2 - 1,$$
$$Q_{nm} = 0, \quad \text{for} \quad m - n \geqslant 2, \quad n - m \geqslant p_2, \tag{76}$$
$$\bar{Q}_{nn} = b_0(n), \qquad \bar{Q}_{n,n-1} = R_n, \qquad \bar{Q}_{n,n+k} = b_k(n) R^{-1}_{n+1} \ldots R^{-1}_{n+k}, \qquad k = 1, \ldots, p_1 - 1,$$
$$\bar{Q}_{nm} = 0, \quad \text{for} \quad n - m \geqslant 2, \quad m - n \geqslant p_1. \tag{77}$$

The subscripts $-/+$ in (73), (74) denote lower/upper triangular parts, whereas $(+)/(-)$ denote upper/lower triangular plus diagonal parts. In the case under consideration the number $p_2 = 3$ in (76), whereas the number p_1 in (77) is arbitrary finite or ∞ [6].

Note the presence of the nonevolution constraint equation (75), which is called "string equation". The lattice equations for the matrix elements $a_k(n)$ of Q (the first Eqs. (73) and (74)) can be solved explicitly as functionals of the matrix elements of $\bar{Q}$,

$$Q_{(-)} = \sum_{s=0}^{p_2-1} \alpha_s \bar{Q}^s_{(-)}, \qquad \alpha_s \equiv -(s+1)\frac{\tilde{t}_{s+1}}{g}. \tag{78}$$

Furthermore, it is more convenient to introduce another matrix $\hat{Q}$ (with matrix elements $\hat{R}_n, \hat{b}_k(n)$, cf. (77)) in place of $\bar{Q}$ defined as

$$\hat{Q}^{p_2-1}_{(-)} = \sum_{s=0}^{p_2-1} \alpha_s \bar{Q}^s_{(-)}, \qquad \hat{Q}^{p_2-1}_{(+)} = I_+ \longrightarrow \hat{Q}^{p_2-1} = Q, \tag{79}$$

[6] Both numbers $p_{1,2}$ indicating the number of nonzero diagonals, outside the main one, of the matrices $\bar{Q}$ and Q are related with the polynomial orders of the corresponding string two-matrix model potentials, whereas the constant g in (75) denotes the coupling parameter between the two random matrices.

with $I_{+nm} = \delta_{n+1,m}$, where the last equality follows from (78). More generally, we have the relations $\hat{Q}^s_{(-)} = \sum_{\sigma=0}^{s} \gamma_{s\sigma}\bar{Q}^{\sigma}_{(-)}$ for any $s = 1,\dots,p_2$ with coefficients $\gamma_{s\sigma}$ simply expressed through α_s (78). Specifically we obtain

$$\hat{R}_n = \gamma_{11}R_n, \qquad \hat{b}_0(n) = \gamma_{11}b_0(n) + \frac{\gamma_{21}}{2\gamma_{11}}, \qquad \hat{b}_1(n) = \gamma_{11}^2 b_1(n) + \frac{\gamma_{31}}{3\gamma_{11}} - (\frac{\gamma_{21}}{2\gamma_{11}})^2, \tag{80}$$

$$\gamma_{11} \equiv (\alpha_{p_2-1})^{1/(p_2-1)}, \qquad \gamma_{21} \equiv \frac{2}{p_2-1}\frac{\alpha_{p_2-2}}{(\alpha_{p_2-1})^{(p_2-3)/(p_2-1)}},$$

$$\gamma_{31} \equiv \frac{3}{p_2-1}\left(\frac{\alpha_{p_2-3}}{(\alpha_{p_2-1})^{(p_2-4)/(p_2-1)}} - \frac{p_2-4}{2(p_2-1)}\frac{\alpha_{p_2-2}^2}{(\alpha_{p_2-1})^{(2p_2-5)/(p_2-1)}}\right). \tag{81}$$

Accordingly, the evolution equations (74) acquire the form

$$\frac{\partial}{\partial \hat{t}_s}\hat{Q} = [\hat{Q}, \hat{Q}^s_-], \qquad \text{with } \frac{\partial}{\partial \hat{t}_s} = \sum_{\sigma=1}^{s} \gamma_{s\sigma}\frac{\partial}{\partial \bar{t}_\sigma}. \tag{82}$$

The remaining independent lattice equations then read (we write down explicitly only the $\hat{t}_1 \equiv x$ and $\hat{t}_2$ evolution equations for the $p_2 = 3$ case),

$$\partial_x \ln \hat{R}_{n+1} = \hat{b}_0(n+1) - \hat{b}_0(n), \qquad \hat{b}_1(n) - \hat{b}_1(n-1) = \partial_x \hat{b}_0(n), \tag{83}$$

$$\hat{R}_{n+1} - \hat{R}_n = \partial_x[\hat{b}_0^2(n) + \hat{b}_1(n) + \hat{b}_1(n-1)], \tag{84}$$

$$\frac{\partial}{\partial \hat{t}_2}\hat{R}_{n+1} = \partial_x(\partial_x \hat{R}_{n+1} + 2\hat{b}_0(n)\hat{R}_{n+1}), \tag{85}$$

$$\frac{\partial}{\partial \hat{t}_2}\hat{b}_0(n) = \partial_x[2\hat{b}_1(n) + \hat{b}_0^2(n) - \partial_x \hat{b}_0(n)], \qquad \frac{\partial}{\partial \hat{t}_2}\hat{b}_1(n) = \partial_x \hat{R}_{n+1}. \tag{86}$$

Now, we observe that the system of Darboux-Bäcklund equations for SL(3,1) KP-KdV hierarchy (68)-(70) exactly coincides upon identification,

$$B^{(n)} = \hat{b}_0(n-1), \qquad u^{(n)} = 2\hat{b}_1(n-1), \qquad A^{(n)} = \hat{R}_n, \tag{87}$$

with the $x \equiv \hat{t}_1$ constrained Toda lattice evolution equations (83), (84). Also, the higher Toda lattice evolution parameters can be identified with the following subset of evolution parameters of the SL$(p_2, 1)$ KP-KdV hierarchy (63) [11,9],

$$\hat{t}_s \simeq t_s^{\text{KP-KdV}}, \qquad s = 2,\dots,p_2, \qquad t_r \simeq t_{r(p_2-1)}^{\text{KP-KdV}}, \qquad r = 1,\dots,p_1, \tag{88}$$

the second identification resulting from (79).

In particular, excluding $\hat{b}_0(n) \equiv B^{(n+1)}$ and $\hat{b}_1(n) \equiv \frac{1}{2}u^{(n+1)}$ in (70) using (83)-(86), we obtain the two-dimensional Toda lattice equation for $A^{(n)} \equiv \hat{R}_n$,

$$\partial_x \partial_{\hat{t}_2} \ln A^{(n)} = A^{(n+1)} - 2A^{(n)} + A^{(n-1)}. \tag{89}$$

6. Discussion and outlook

6.1. Partition function of the two-matrix string model

The partition function Z_N of the two-matrix string model is simply expressed in terms of the $\bar{Q}$ matrix element $b_1(N-1)$ at the Toda lattice site $N-1$, where N indicates the size $(N \times N)$ of the pertinent random matrices:

$\partial_x^2 \ln Z_N = b_1(N-1)$ (cf. Refs. [4,11]). Thus, using (65) and (57) together with (88), and accounting for the relations (80), (81), we obtain the following exact solution at *finite* N for the two-matrix model partition function,

$$Z_N = W[\hat{\Phi}^{(0)}, \partial_x^{p_2-1}\hat{\Phi}^{(0)}, \ldots, \partial_x^{(p_2-1)(N-1)}\hat{\Phi}^{(0)}] \exp\left\{ \int^{\hat{t}_1}\!\!\int^{\hat{t}_1} \left[\left(\frac{\gamma_{21}}{2\gamma_{11}} \right)^2 - \frac{\gamma_{31}}{3\gamma_{11}} \right] \right\}, \tag{90}$$

$$\hat{\Phi}^{(0)} = \int_\Gamma \frac{\mathrm{d}\lambda}{2\pi} c(\lambda) \exp[\hat{\xi}(\lambda, \{\hat{t}, t\})], \qquad \hat{\xi}(\lambda, \{\hat{t}, t\}) \equiv \sum_{s=1}^{p_2} \lambda^s \hat{t}_s + \sum_{r=2}^{p_1} \lambda^{r(p_2-1)} t_r, \tag{91}$$

where $x \equiv \hat{t}_1$ and the γ-coefficients are defined in (81). The "density" function $c(\lambda)$ in (91) is determined from matching the expression for $\hat{\Phi}^{(0)}$: $\partial_x \ln \hat{\Phi}^{(0)} = \hat{b}_0(0) = \gamma_{11} b_0(0) + \gamma_{21}/2\gamma_{11}$ (cf. (71), (87), (80)), with the expression for $b_0(0)$ in the orthogonal-polynomial formalism [4],

$$\int_\Gamma \frac{\mathrm{d}\lambda}{2\pi} c(\lambda) \exp\left(\sum_{s=1}^{p_2} \lambda^s \hat{t}_s + \sum_{r=2}^{p_1} \lambda^{r(p_2-1)} t_r \right)$$
$$= \exp\left(\int^{\tilde{t}_1} \frac{\gamma_{21}}{2\gamma_{11}^2} \right) \int_\Gamma\!\!\int_\Gamma \mathrm{d}\lambda_1\, \mathrm{d}\lambda_2 \exp\left(\sum_{r=1}^{p_1} \lambda_1^r t_r + \sum_{s=1}^{p_2} \lambda_2^s \tilde{t}_s + g\lambda_1\lambda_2 \right)\Bigg|_{t_1 = \hat{t}_{p_2-1}(\hat{t})}. \tag{92}$$

Obviously, the most important question now is to study the physical *double-scaling* limit [3] of (90), which amounts to a special fine-tuned limit $N \to \infty$. The latter presumably includes renormalizations and critical point approaching of the $(t_r, \tilde{t}_s)$ parameters.

6.2. Connection to Grassmannian manifolds and n-soliton solution for the KP hierarchy

Let $\{\psi_1, \ldots, \psi_n\}$ be a basis of solutions of the nth order equation $L\psi = 0$, where $L = (D+v_n)(D+v_{n-1})\ldots \times (D+v_1)$. If W_k denotes the Wronskian determinant of $\{\psi_1, \ldots, \psi_k\}$ then one can show that [16,18]

$$v_i = \partial\left(\ln \frac{W_{i-1}}{W_i} \right), \qquad W_0 = 1. \tag{93}$$

This allows to show that the space of differential operators is parametrized by the Grassmannian manifold (see e.g. Refs. [18,19]). Start namely with the given differential operator $L_n = D^n + u_1 D^{n-1} + \ldots + u_n$ and determine the kernel of L_n given by an n-dimensional subspace of some Hilbert space of functions $\mathcal{H}$, spanned, let us say, by $\{\psi_1, \ldots, \psi_n\}$. This establishes the connection one way. On the other hand let $\{\psi_1, \ldots, \psi_n\}$ be a basis of one point Ω of $\mathrm{Gr}^{(n)}$ being a Grassmannian manifold. Define the differential equation as $L_n(\Omega) f = W_k(f)/W_k$. From (23) this associates the differential operator

$$L_n = D^n + u_1 D^{n-1} + \ldots + u_n = (D+v_n)(D+v_{n-1})\ldots(D+v_1), \tag{94}$$

given by a Miura correspondence to a given point of the Grassmannian.

Recall now the correspondence (equivalence) between N-generalized two-boson KP and $2N$-boson KP systems, Eqs. (37)–(40). Relation (40) has the form like in (23) and, therefore

$$\{[(D - B_i^{(N)})^{-1}(D - B_{i+1}^{(N)})^{-1} \ldots (D - B_N^{(N)})^{-1}]^{-1}\}^\dagger \Psi_j = 0, \qquad i \leqslant j \leqslant N. \tag{95}$$

The relations above generalize the relations encountered in the study of flags manifolds and clearly deserve further investigations.

Let us now comment on the connection to the n-soliton solution for the KP hierarchy. Assume that the above functions ψ_i, $i = 1, \ldots, n$, have the property $\partial_m \psi_i = \partial^m \psi_i$ for arbitrary $m \geqslant 1$ $(\partial_m \equiv \partial/\partial t_m)$, in other words ψ_i are eigenfunctions of $L^{(0)} = D$. We introduce $L \equiv L_n D L_n^{-1}$, where L_n is defined in terms of $\{\psi_1, \ldots, \psi_n\}$ as in (93) and (94). It is known that such a Lax operator satisfies a generalized Lax equation $\partial_m L = [L_+^m, L]$ [1,20].

Using (21) we can rewrite the above Lax operator as a result of successive DB transformations applied to D,

$$L = L_n D L_n^{-1} = T_n T_{n-1} \ldots T_1 D T_1^{-1} \ldots T_{n-1}^{-1} T_n^{-1}, \tag{96}$$

where the T_i are given in terms of Wronskians as in (22). It follows that L can be cast in the form of the Lax operator belonging to the n-generalized two-boson KP hierarchy having the form as in (1) with $r = 1, N = n$. Using the formalism developed in this paper one can prove by induction that the corresponding τ-function of L takes a Wronskian form $\tau_n = W_n[\psi_1, \ldots, \psi_n]$ reproducing the n-soliton solution to the KP equation derived in Ref. [21]. In fact, choosing $\psi_i = \exp(\sum t_k \alpha_i^k) + \exp(\sum t_k \beta_i^k)$ allows one to rewrite τ_n in the conventional form of the n-soliton solution to the KP equation [22].

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Note added

After completion of this paper we became aware of Ref. [23] where Wronskian expressions for partition functions of matrix models have been obtained by a different method. We would like to stress that our result (90)–(92) explicitly incorporates the “string-equation” constraint (75) on the Toda lattice system.

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